TRIHARMONIC SELF-OSCILLATIONS IN COUETTE–POISEUILLE FLOW

A. T. Zinov'ev and A. M. Lifshits UDC 532.517.4

One of the fundamental problems in the theory of hydrodynamical stability is the determination of the Reynolds number for the transition between laminar and turbulent flow. The most difficult case is a rigid loss of stability, where the linear theory gives only an upper limit below which the laminar regime can be prolonged under ideal conditions. A typical example is flow in a plane channel. For Poiseuille flow, the critical Reynolds number, constructed from the mean velocity and the half-width of the channel, is $R_L = 3848$ according to the linear theory [1], whereas turbulence is observed for Reynolds numbers around 700 [2]. For Couette–Poiseuille flow with a velocity profile $U_L = (3/2)(1 - |q|)(1 - y^2) + qy$ the Reynolds number in the linear theory goes to infinity when $|q| \to 0.26$ [3, 4], but in Couette flow ($q = 1$) the transition to turbulence takes place for $600 < R < 10^4$ [5, 6].

1. Experiments indicate that one of the basic mechanisms of the transition in boundary layers is a secondary instability of two-dimensional Tollmien–Schlichting waves against three-dimensional perturbations of half the frequency [7-9]. The simplest case is that of a symmetric triplet, a resonant group of three waves consisting of one plane wave with frequency ω and longitudinal wave number α and a pair of three-dimensional waves with parameters ($\omega/2$, $\alpha/2$, $\pm\beta$), where β is the transverse wave number.

The importance of three-dimensional perturbations for the transition in plane Couette and Poiseuille flows was pointed out in [10]. It was noted that the two-dimensional self-oscillation in Couette flow is unstable. A simple model which takes into account the strong nonlinear interaction is the representation of the three-dimensional perturbations by a symmetric triplet. The critical stationary flow regime for Poiseuille flow was calculated in [11] with this approach. The results not only differ strongly from the case of two-dimensional self-oscillations, but are in good agreement with experiment. Below we present the results of an analogous calculation for Couette–Poiseuille flow.

In [1] an escalator scheme for analyzing hydrodynamic stability was proposed, based on successively taking into account the fundamental interactions of the perturbations. The triplet model is the next step, following the monoharmonic model. In this case the fluctuation part of the velocity is represented in the form

$$v_k = 2\,\mathrm{Re}\left\{J_k \sum_{j=1}^{3} v_{kj}(y) \exp\left[i\left(\alpha_j x + \beta_j z - \alpha_j C_j t\right)\right]\right\},$$

where α_j and β_j are the longitudinal and transverse wave numbers; C_j are the phase velocities; J_k are amplitude factors dependent on the normalization conditions; and y is the transverse coordinate. The parameters of the carrier harmonics obey the resonance relations $\sum_{j=1}^{3} \alpha_j = 0$, $\sum_{j=1}^{3} \beta_j = 0$, $\sum_{j=1}^{3} \omega_j = 0$, where the frequencies $\omega_j = \alpha_j C_j$.

Equations for the amplitudes u, v, w and the average velocity profile are obtained after projection onto the base and zero harmonics. For a symmetric triplet, the resonance condition leads to the simple form $\alpha_1 = \alpha$, $\beta_1 = 0$, $\alpha_2 = \alpha_3 = -\alpha/2$, $\beta_2 = -\beta_3 = \beta$, $C_1 = C_2 = C_3 = C$, and $w_1 = 0$, $u_2 = u_3$, $v_2 = v_3$, $w_2 = -w_3$.

After carrying out the projection we obtain

$$L_1 u_1 + U' v_1 + i\alpha_1 p_1 = -2\left(J_2^2/J_1\right)\left[i\alpha_1 \bar{u}_2^2 + (\bar{u}_2 \bar{v}_2)'\right],$$
$$L_1 v_1 + p_1' = -2\left(J_2^2/J_1\right)\left[i\alpha_1 \bar{v}_2 \bar{u}_2 + (\bar{v}_2^2)'\right], \quad i\alpha_1 u_1 + v_1' = 0, \tag{1.1}$$

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$$L_2u_2 + U'v_2 + i\alpha_2 p_2 = -J_1 [2i\alpha_2 \overline{u}_2\overline{u}_1 + (\overline{u}_2\overline{v}_1 + \overline{u}_1\overline{v}_2)' + i\beta_2\overline{w}_2\overline{u}_1], \quad (1.1)$$

$$L_2v_2 + p_2' = -J_1 [i\alpha_2 (\overline{v}_2\overline{u}_1 + \overline{v}_1\overline{u}_2) + 2(\overline{v}_1\overline{v}_2)' + i\beta_2\overline{w}_2\overline{v}_1],$$

$$L_2w_2 + i\beta_2 p_2 = -J_1 [i\alpha_2\overline{w}_2\overline{u}_1 + (\overline{w}_2\overline{v}_1)'], \quad i\alpha_2 u_2 + v_2' + i\beta_2 w_2 = 0,$$

$$U_д'' = R_p [\tau_1' + 2\tau_2'], \quad L_j = -\frac{1}{R_p}\left(\frac{\partial^2}{\partial y^2} - k_j^2\right) + i\alpha_j (U - C), \quad k_j^2 = \alpha_j^2 + \beta_j^2,$$

where $\tau_j = J^2{}_j(u_j\bar{v}_j + \bar{u}_j v_j)$ is the contribution of the j-th harmonic to the Reynolds stress. The average velocity U is the sum of the laminar velocity and an additional velocity: $U = U_L + U_{add}$. For a known R_p, we can construct a Reynolds number from the mean flow velocity $R = R_p\langle U\rangle$ and from the velocity of the wall $R_c = R_p|q|$.

The system (1.1), when supplemented by the adherence conditions, forms an eigenvalue problem. Of the parameters (α, β, R_p, C), three are specified arbitrarily (within definite limits), and one is sought as the eigenvalue. The system (1.1) has a certain level of degeneracy: first the laminar limit, where all fluctuation amplitudes are zero; second, the monoharmonic limit, when only the two-dimensional harmonics have nonzero amplitude; third, when the amplitudes of all harmonics are nonzero, but the quantity C is far from the eigenvalues of the operator defined by the left-hand side of the first three equations of (1.1). In this case, J_1 is of order $J^2{}_2$ and the two-dimensional waves weakly affect the nature of the solution. If C is close to an eigenvalue (in the sense discussed above), then the right-hand sides of the first two equations of (1.1) play the role of applied forces acting at the resonance frequency. The quantity J_1 disproportionately increases, which in turn has an important "inverse" effect on the characteristics of the solution. This situation occurs when the triplet is in resonance.

System (1.1) was solved numerically by Newton's method with a second-order approximation on a nonuniform grid with a sinusoidal compression toward the walls. The number of grid points N = 100. A manifold of solutions at a single point corresponds to neutral curves of the linear theory for those q for which it exists. For q = 0, the bifurcation point has the parameters [1]:

$$R_p = 7776, \ \alpha = 1.088, \ \beta = 0.709, \ C = 0.358 \quad (2.1)$$

and the amplitudes in this case are the eigenfunctions of the linear problem with zero intensity. Because Newton's method requires a good initial approximation, the parameters were varied continuously, starting from (2.1).

At the point (2.1), the amplitudes of all harmonics are symmetric with respect to the transverse component of the velocity. However, it is easy to see that for finite intensities this type of solution is impossible for a symmetric profile, i.e., the solution for Poiseuille flow is asymmetric. An asymmetry of secondary flow with symmetric boundary conditions is not an exceptional occurrence in hydrodynamics. It occurs, for example, for flow in a diffuser and in the streamlining of a symmetric body (Karman effect). Asymmetry of the perturbations in Poiseuille flow has been observed experimentally [7-9], and although the profile of the average nonstationary flow is symmetric, sometimes trajectories can be found close to solutions with various asymmetries. However, it must be remembered that when q = 0 the asymmetric solutions can be transformed into each other by changing the sign of y. For a nonzero q, there is no such relation between the solutions. Consequently, when $q \neq 0$ there exist two different branches of the solution. A pair of solutions transforms into each other if we simultaneously change the signs of y and q. Hence it is sufficient to calculate both branches for q > 0 or one branch for all q. The latter possibility is the most convenient, and the one used here.

The numerical results support the asymmetry with respect to q of the characteristics of the critical self-oscillations. In Fig. 1 we give $R_{p*}(q)$ and $\alpha_* R_{p*}(q)$ on the leading edges of the neutral hypersurfaces. It is clear that the solutions exist in the region $|q| \leqslant 0.62$. Solutions skewed toward the wall and moving in the same direction as the perturbation exist for a wider region ($|q| \leqslant 0.62$). For solutions with the opposite asymmetry, the critical Reynolds number increases rapidly and goes to infinity for $|q| \approx 0.45$. For $|q| > 0.26$, the nonlinear neutral surfaces, as though suspended above the surface of zero intensity, do not touch it anywhere.

Figure 2 shows the critical values of the parameters of the harmonics for different values of q. Note that α_* and β_* depend strongly on the sign of q. It is interesting that

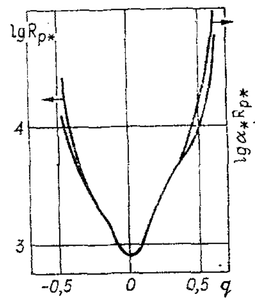

Fig. 1

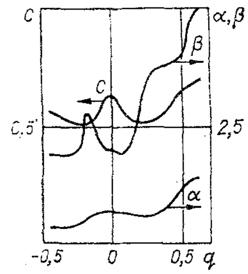

Fig. 2

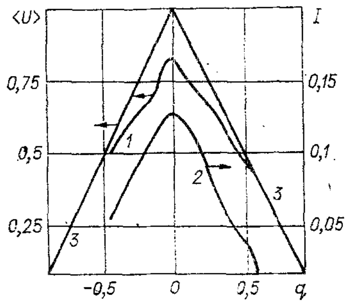

Fig. 3

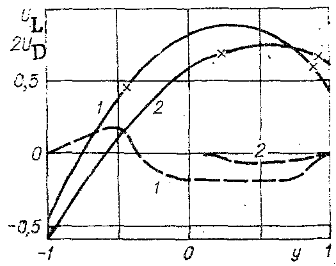

Fig. 4

when $R_p{*} \to \infty$, $\alpha{*}$ goes to a finite limit, whereas in the linear case the critical R_p and α are related in such a way that $\alpha{*} \to 0$ when $R_p{*} \to \infty$ and the product $\alpha{*}R_p{*}$ remains finite.

Curves 1-3 of Fig. 3 show the dependence on q of the integral characteristics: the mean velocity of flow <U>, the perturbation intensity $I = \left(\frac{1}{2}\int_{-1}^{1}\langle v_k v_k\rangle\, dy\right)^{1/2} \Big/ \langle U\rangle$, and the mean velocity of laminar flow, respectively. It is seen that with increasing $|q|$, deformation of the mean velocity profile rapidly decreases. The perturbation intensity also decreases, but much more slowly. It can be assumed that asymptotically (when $R_p{*} \to \infty$) the Reynolds stress is equal to zero even though the perturbation has a finite amplitude.

This assumption is to some extent supported by the curves of Fig. 4. The dashed curves 1 and 2 show the nature of the deformation of the mean velocity profile $U_D = U - U_L$ for different values of q: 1 corresponds to q = −0.45, and 2 to q = −0.62. Note the rapid decrease of the velocity profile deformation and the localization of the Reynolds stresses near one of the walls of the channel. The solid curves 1 and 2 refer to the laminar velocity profiles for the same values of q; the crosses show the positions of the critical layers y_c for which $U(y_c) = C$. We note the tendency of one of the critical layers to approach the wall on the existence boundary of the solution.

Therefore, the region of existence of the three-dimensional self-oscillations of finite amplitude considered here does not include Couette flow. Apparently this is due to the fact that the most critical perturbations for Couette flow have a different structure from those for Poiseuille flow.

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CLASS OF SELF-SIMILAR SOLUTIONS FOR A HIGH-TEMPERATURE AXISYMMETRIC JET

A. A. Bobnev

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The most simple and rigorous results in the investigation of nonisothermal jet flows of a compressible gas can be obtained by utilizing the Dorodnitsyn transformation [1]. However, this method is suitable only for plane (or nearly plane) gas flows with a linear dependence of the heat conduction and dynamic viscosity on temperature; the transition here from Dorodnitsyn to physical variables is difficult. In the case of an axisymmetric jet issuing from a point source for a domain where the temperature on the axis is considerably higher than the temperature at infinity, by using the idea of the existence of a separating layer [2], a self-similar solution can be constructed for a power-law dependence of the heat conduction and viscosity on the temperature, where it is possible to go from the initial two-parameter problem (the Prandtl number, the exponent) to a one-parameter problem.

1. We write the problem describing the emergence of a nonisothermal jet from a cylindrical orifice in the boundary-layer approximation in the dimensionless form

$$\frac{1}{r}\frac{\partial}{\partial r}\left[r\mu(T)\frac{\partial w}{\partial r}\right]=\rho\left(v\frac{\partial w}{\partial r}+w\frac{\partial w}{\partial z}\right), \tag{1.1}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(r\rho v)+\frac{\partial}{\partial z}(\rho w)=0,\quad \rho T=1,$$
$$\frac{1}{r}\frac{\partial}{\partial r}\left[r\lambda(T)\frac{\partial T}{\partial r}\right]=\Pr\rho\left(v\frac{\partial T}{\partial r}+w\frac{\partial T}{\partial z}\right);$$
$$v=\frac{\partial w}{\partial r}=\frac{\partial T}{\partial r}=0 \quad \text{for } r=0; \tag{1.2}$$

$$T=\varepsilon,\ w=0 \quad \text{for } r\to\infty, \tag{1.3}$$

where r, zR are cylindrical coordinates (r, z are internal coordinates in an asymptotic expansion in the small parameter R^{-1}), $R=\sqrt{\rho_m I_{1m}/2\pi}/\mu_m$ is a certain analog of the Reynolds number, vR^{-1}, w – r, z are velocity components; $\Pr = c_{pm}\mu_m/\lambda_m$ is the Prandtl number; and ε is the value of the temperature at infinity. The notation of the remaining quantities is standard. The scales T_m, ρ_m, c_{pm}, μ_m, λ_m (the scale quantities are marked with the subscript m), as well as the total momentum scale I_{1m} are the enthalpy flux I_{2m} defined by the formulas

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